A Hardy's Uncertainty Principle Lemma in Weak Commutation Relations of Heisenberg-Lie Algebra

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Abstract. In this article we consider linear operators satisfying a generalized commutation relation of a type of the Heisenberg-Lie algebra. It is proven that a generalized inequality of the Hardy's uncertainty principle lemma follows. Its applications to time operators and abstract Dirac operators are also investigated.

Key words : weak commutation relations, Heisenberg-Lie algebra, time operators, Hamiltonians, time-energy uncertainty relation, Dirac operators, essential self-adjointness.
MSC 2010 : 81Q10, 47B25, 46L60.

1 Introduction and Results

In this article we investigate a norm-inequality of the linear operators which obey a generalized weak commutation relation of a type of the Heisenberg-Lie algebra, and consider its application to the theory of the time operator [7, 2], and an abstract Dirac operator. Let $\mathbf{X} = \{X_j\}_{j=1}^N$, $\mathbf{Y} = \{Y_j\}_{j=1}^N$ and $\mathbf{Z} = \{Z_j\}_{j=1}^N$ be symmetric operators on a Hilbert space $\mathcal{H}$. The weak commutator of operators A and B is defined for $\psi \in \mathcal{D}(A) \cap \mathcal{D}(B)$ and $\phi \in \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$ by

$$[A,B]^{\mathrm{w}}(\phi\,,\,\psi) \;=\; (A^*\phi\,,B\psi) \,-\, (B^*\phi,A\psi).$$

Here the inner product has a linearity of $(\eta, \alpha\psi + \beta\phi) = \alpha(\eta,\psi) + \beta(\eta,\phi)$ for $\alpha, \beta \in \mathbf{C}$. We assume that $(\mathbf{X},\mathbf{Y},\mathbf{Z})$ satisfies the following conditions.

(A.1) Z_j, $1 \le j \le N$, is bounded operator.

(A.2) Let $\mathcal{D}_{\mathbf{X}} = \cap_{j=1}^N \mathcal{D}(X_j)$ and $\mathcal{D}_{\mathbf{Y}} = \cap_{j=1}^N \mathcal{D}(Y_j)$. It follows that for $\phi, \psi \in \mathcal{D}_{\mathbf{X}} \cap \mathcal{D}_{\mathbf{Y}}$,

$$\begin{aligned}
&[X_j, Y_l]^{\mathrm{w}}(\phi\,,\psi) \;=\; \delta_{j,l}\,(\phi,\, iZ_j\psi),\\
&[X_j, Z_l]^{\mathrm{w}}(\phi\,,\psi) \;=\; [Y_j, Z_l]^{\mathrm{w}}(\phi\,,\psi) \;=\; 0\\
&[X_j, X_l]^{\mathrm{w}}(\phi\,,\psi) \;=\; [Y_j, Y_l]^{\mathrm{w}}(\phi\,,\psi) \;=\; [Z_j, Z_l]^{\mathrm{w}}(\phi\,,\psi) \;=\; 0.
\end{aligned}$$

Note that $[Z_j, Z_l]\psi = 0$ follows for $\psi \in \mathcal{H}$, since Z_j, $j = 1, \cdots, N$, is bounded. In this article we consider an generalization of the inequality

$$\int_{\mathbf{R}^N} \frac{1}{|\mathbf{r}|^2}|u(\mathbf{r})|^2 d\mathbf{r} \;\le\; \frac{4}{(N-2)^2}\int_{\mathbf{R}^N} |\nabla u(\mathbf{r})|^2\, d\mathbf{r}, \qquad\qquad N \ge 3.$$

This inequality is a basic one of Hardy's uncertainty principle inequalities. For Hardy's uncertainty inequalities, refer to e.g. [5, 6, 13].

Let us introduce the additional conditions.

(A.3) X_j is self-adjoint for all $1 \leq j \leq N$.

(A.4) X_i and Z_l strongly commutes for all $1 \leq j \leq N$ and $1 \leq l \leq N$.

Sicne $Z_j,\ j = 1, \cdots, N$, is bounded self-adjoint operator, we can set $\lambda_{\min}(\mathbf{Z})$ and $\lambda_{\max}(\mathbf{Z})$ by

$$\begin{aligned} \lambda_{\min}(\mathbf{Z}) &= \min_{1 \leq j \leq N} \inf \sigma(Z_j), \\ \lambda_{\max}(\mathbf{Z}) &= \max_{1 \leq j \leq N} \sup \sigma(Z_j), \end{aligned}$$

where $\sigma(O)$ denotes the spectrum of the operator O.

Theorem 1 *Assume* ***(A.1)-(A.4)****. Let* $\Psi \in \mathcal{D}(|\mathbf{X}|^{-1}) \cap \mathcal{D}_{\mathbf{X}} \cap \mathcal{D}_{\mathbf{Y}}$*. Then the following* ***(1)*** *and* ***(2)*** *hold*
(1) *If* $N\lambda_{min}(\mathbf{Z}) - 2\lambda_{max}(\mathbf{Z}) > 0$*, it follows that*

$$\left\| \, |\mathbf{X}|^{-1}\Psi \right\|^2 \leq \frac{4}{(N\lambda_{min}(\mathbf{Z}) - 2\lambda_{max}(\mathbf{Z}))^2} \sum_{j=1}^{N} \left\| Y_j \Psi \right\|^2 . \tag{1}$$

(2) *If* $2\lambda_{min}(\mathbf{Z}) - N\lambda_{max}(\mathbf{Z}) > 0$*, it follows that*

$$\left\| \, |\mathbf{X}|^{-1}\Psi \right\| \leq \frac{4}{(2\lambda_{min}(\mathbf{Z}) - N\lambda_{max}(\mathbf{Z}))^2} \sum_{j=1}^{N} \left\| Y_j \Psi \right\|^2 . \tag{2}$$

Before proving Theorem 1, let us consider the replacement of $\mathbf{X}$ and $\mathbf{Y}$ in Theorem 1. Let us introduce the following conditions substitute for **(A.3)** and **(A.4)**.

(A.5) Y_j is self-adjoint for all $1 \leq j \leq N$.

(A.6) Y_i and Z_l strongly commutes for all $1 \leq j \leq N$ and $1 \leq l \leq N$.

It is seen from **(A.2)**, that

$$[Y_j, X_l]^{\mathrm{W}}(\phi, \psi) = \delta_{j,l}(\phi, i(-Z_j)\psi), \qquad \phi, \psi \in \mathcal{D}_{\mathbf{X}} \cap \mathcal{D}_{\mathbf{Y}}. \tag{3}$$

Note that $\inf \sigma(-Z_j) = -\sup(Z_j)$ and $\sup(-Z_j) = -\inf \sigma(Z_j)$ follow. Then we obtain a following corollary :

Corollary 2 *Assume* ***(A.1)-(A.2)*** *and* ***(A.5)-(A.6)****. Let* $\Psi \in \mathcal{D}(|\mathbf{Y}|^{-1}) \cap \mathcal{D}_{\mathbf{X}} \cap \mathcal{D}_{\mathbf{Y}}$*. Then the following* ***(1)*** *and* ***(2)*** *hold.*
(1) *If* $2\lambda_{min}(\mathbf{Z}) - N\lambda_{max}(\mathbf{Z}) > 0$*, it follows that*

$$\left\| \, |\mathbf{Y}|^{-1}\Psi \right\| \leq \frac{4}{(2\lambda_{min}(\mathbf{Z}) - N\lambda_{max}(\mathbf{Z}))^2} \sum_{j=1}^{N} \left\| X_j \Psi \right\|^2 . \tag{4}$$

(2) *If* $N\lambda_{min}(\mathbf{Z}) - 2\lambda_{max}(\mathbf{Z}) > 0$*, it follows that*

$$\left\| \, |\mathbf{Y}|^{-1}\Psi \right\| \leq \frac{4}{(N\lambda_{min}(\mathbf{Z}) - 2\lambda_{max}(\mathbf{Z}))^2} \sum_{j=1}^{N} \left\| X_j \Psi \right\|^2 . \tag{5}$$

(Proof of Theorem 1)

(1) Let $\Psi \in \mathcal{D}(|\mathbf{X}|^{-1}) \cap \mathcal{D}_{\mathbf{X}} \cap \mathcal{D}_{\mathbf{Y}}$. For $\varepsilon > 0$ and $t > 0$, it is seen that

$$\left\| \left(Y_j - itX_j(\mathbf{X}^2+\varepsilon)^{-1}\right) \Psi \right\|^2 = \|Y_j\Psi\|^2 - it\,[Y_j,\, X_j(\mathbf{X}^2+\varepsilon)^{-1}]^{\mathrm{w}}(\Psi,\Psi) + t^2 \left\|X_j(\mathbf{X}^2+\varepsilon)^{-1}\Psi\right\|^2 . \tag{6}$$

We see that

$$[Y_j,\, X_j(\mathbf{X}^2+\varepsilon)^{-1}]^{\mathrm{w}}(\Psi,\Psi) = [Y_j, X_j]^{\mathrm{w}}(\Psi,\, (\mathbf{X}^2+\varepsilon)^{-1}\Psi) + [Y_j,\, (\mathbf{X}^2+\varepsilon)^{-1}]^{\mathrm{w}}(X_j\Psi,\Psi). \tag{7}$$

From **(A.2)** and **(A.4)**, we obtain that

$$[Y_j, X_j]^{\mathrm{w}}(\Psi,\, (\mathbf{X}^2+\varepsilon)^{-1}\Psi) = -i\,((\mathbf{X}^2+\varepsilon)^{-1/2}\Psi,\, Z_j(\mathbf{X}^2+\varepsilon)^{-1/2}\Psi). \tag{8}$$

Note that for a symmetric operator A and the non-negative symmetric operator B, the resolvent formula $[A,(B+\lambda)^{-1}]^{\mathrm{w}}(v,u) = [B,A]^{\mathrm{w}}((B+\lambda)^{-1}v,\, (B+\lambda)^{-1}u)$ for $\lambda > 0$ follows. Then by using this formura, **(A.2)** and **(A.4)** yield that

$$[Y_j,\, (\mathbf{X}^2+\varepsilon)^{-1}]^{\mathrm{w}}(X_j\Psi,\Psi) = 2i(X_j(\mathbf{X}^2+\varepsilon)^{-1}u\,,\, Z_jX_j(\mathbf{X}^2+\varepsilon)^{-1}u) \tag{9}$$

Since $\left\|\left(Y_j - itX_j(\mathbf{X}^2+\varepsilon)^{-1}\right) u\right\|^2 \geq 0$ and $t > 0$, we see from (7), (8) and (9) that

$$\begin{aligned}
&\|Y_j\Psi\|^2 \\
&\geq -t^2 \left\|X_j(\mathbf{X}^2+\varepsilon)^{-1}u\right\|^2 + t\,((\mathbf{X}^2+\varepsilon)^{-1/2}\Psi,\, Z_j(\mathbf{X}^2+\varepsilon)^{-1/2}u) - 2t(X_j(\mathbf{X}^2+\varepsilon)^{-1}u\,,\, Z_jX_j(\mathbf{X}^2+\varepsilon)^{-1}\Psi) \\
&\geq \Big(-t^2 - 2t\lambda_{\max}(\mathbf{Z})\Big) \left\|X_j(\mathbf{X}^2+\varepsilon)^{-1}u\right\|^2 + t\lambda_{\min}(\mathbf{Z})\|(\mathbf{X}^2+\varepsilon)^{-1/2}\Psi\|. \qquad (10)
\end{aligned}$$

Then we have that

$$\sum_{j=1}^{N} \|Y_j\Psi\|^2 \geq \Big(-t^2 - 2t\lambda_{\max}(\mathbf{Z})\Big) \left\||\mathbf{X}|(\mathbf{X}^2+\varepsilon)^{-1}\Psi\right\|^2 + tN\lambda_{\min}(\mathbf{Z})\|(\mathbf{X}^2+\varepsilon)^{-1/2}\Psi\|. \tag{11}$$

Note that $\lim_{\varepsilon\to 0} \left\||\mathbf{X}|(\mathbf{X}^2+\varepsilon)^{-1}\Psi\right\|^2 = \||\mathbf{X}|^{-1}\Psi\|$ and $\lim_{\varepsilon\to 0} \|(\mathbf{X}^2+\varepsilon)^{-1/2}\Psi\| = \||\mathbf{X}|^{-1}\Psi\| = 0$ follow from the spectral decomposition theorem. Then we have

$$\sum_{j=1}^{N} \|Y_j\Psi\|^2 \geq \left(-t^2 + (N\lambda_{\min}(\mathbf{Z}) - 2\lambda_{\max}(\mathbf{Z}))t\right) \left\| \, |\mathbf{X}|^{-1}\Psi \right\| . \tag{12}$$

By taking $t = \frac{N\lambda_{\min}(\mathbf{Z})-2\lambda_{\max}(\mathbf{Z})}{2} > 0$ in the right side of (12), we obtain **(1)**.
(2) By computing $\left\|\left(Y_j + itX_j(\mathbf{X}^2+\varepsilon)^{-1}\right)\Psi\right\|^2$ for $t > 0$ and $\varepsilon > 0$, in a similar way of **(1)**, we see that

$$
\begin{aligned}
&\|Y_j\Psi\|^2 \\
&\geq\ -t^2\left\|X_j(\mathbf{X}^2+\varepsilon)^{-1}u\right\|^2\ -t\left((\mathbf{X}^2+\varepsilon)^{-1/2}\Psi, Z_j(\mathbf{X}^2+\varepsilon)^{-1/2}u\right)+2t(X_j(\mathbf{X}^2+\varepsilon)^{-1}u\,, Z_jX_j(\mathbf{X}^2+\varepsilon)^{-1}\Psi) \\
&\geq\ \Big(-t^2+2t\lambda_{\min}(\mathbf{Z})\Big)\left\|X_j(\mathbf{X}^2+\varepsilon)^{-1}u\right\|^2\ -t\lambda_{\max}(\mathbf{Z})\|(\mathbf{X}^2+\varepsilon)^{-1/2}\Psi\|. \qquad (13)
\end{aligned}
$$

Then by taking $\varepsilon \to 0$ in the right side of (13), it follows that

$$
\sum_{j=1}^{N}\|Y_j\Psi\|^2\ \geq\ \Big(-t^2+(\,2\lambda_{\min}(\mathbf{Z})-N\lambda_{\max}(\mathbf{Z}))t\Big)\left\|\,|\mathbf{X}|^{-1}\Psi\right\|. \qquad (14)
$$

By taking $t = \frac{(2\lambda_{\min}(\mathbf{Z})-N\lambda_{\max}(\mathbf{Z}))}{2} > 0$ in (14), we obtain **(2)**. $\blacksquare$.

2 Applications

2.1 Time-Energy Uncertainty inequality

In this subsection we consider an applicaion to the theory of time operators [2, 7]. Let H, T, and C be linear operators on a Hilbert space H. It is said that H has the weak time operator T with the uncommutative factor C if (H,T,C) satisfy the following conditions.

(T.1) H and T are symmetric.

(T.2) C is bounded and self-adjoint.

(T.3) It follows that for $\phi, \psi \in \mathcal{D}(H)\cap\mathcal{D}(T)$,

$$
[T,H]^{\mathrm{w}}(\phi,\psi)\ =\ (\phi, C\psi).
$$

(T.4)

$$
\delta_C\ :=\inf_{\psi\in(\ker C)^{\perp}\setminus\{0\}}\frac{|(\Psi,C\Psi)|}{\|\psi\|^2}\ >\ 0.
$$

Assume that (H,T,C) satisfies **(T.1)-(T.4)**. Then by using $\|Au\|\,\|Bu\| \geq |\mathrm{Im}(Au,Bu)| \geq \frac{1}{2}\left|\,[A,B]^{\mathrm{w}}(u,u)\right|$, it is seen that (H,T,C) satisfies the time-energy uncertainty inequality ([2], Proposition4.1):

$$
\frac{\|\,\left(H-<H>_{\psi}\right)\psi\|\,\|\,\left(T-<T>_{\psi}\right)\psi\|}{\|\psi\|^2}\ \geq\ \frac{\delta_C}{2}, \qquad \psi\in\mathcal{D}(H)\cap\mathcal{D}(T), \qquad (15)
$$

where $<O>_{\psi} =\ (\psi, O\psi)$. From **(2)** in Theorem 1 and **(1)** in Corollary 2, we obtain another type of the inequality between T and H :

Corollary 3 ***(Time-Energy Uncertainty Inequalities)***
Assume ***(T.1)-(T.3)****. Then the following* ***(i)*** *and* ***(ii)*** *hold.*
(i) *If T is self-adjoint, C and T strongly commute, and* $\sup\sigma(C) < 2\inf\sigma(C)$*, it follows that for* $\psi \in \mathcal{D}(|T|^{-1}) \cap \mathcal{D}(T) \cap \mathcal{D}(H)$,

$$\left\| \, |T|^{-1}\psi \right\| \leq \frac{2}{2\inf\sigma(C) - \sup\sigma(C)} \left\| \, H\Psi \right\|. \tag{16}$$

(ii) *If H is self-adjoint, C and H strongly commute, and* $\sup\sigma(C) < 2\inf\sigma(C)$*, it follows that for* $\psi \in \mathcal{D}(|H|^{-1}) \cap \mathcal{D}(H) \cap \mathcal{D}(T)$,

$$\left\| \, |H|^{-1}\psi \right\| \leq \frac{2}{2\inf\sigma(C) - \sup\sigma(C)} \left\| \, T\Psi \right\|. \tag{17}$$

2.2 Abstract Dirac Operators with Coulomb Potential

Next tlt us consider the application to abstract Dirac operators. We consider the self-adjoint operators $\mathbf{P} = \{P_j\}_{j=1}^N$ and $\mathbf{Q} = \{Q_j\}_{j=1}^N$ on a Hilbert space $\mathcal{H}$. Let us set a subspace $\mathcal{D} \subset \cap_{j,l}(\mathcal{D}(P_j) \cap \mathcal{D}(Q_l))$. It is said that $(\mathcal{H}, \mathcal{D}, \mathbf{P}, \mathbf{Q})_N$ is the weak representaion of the CCR with degree N, if $\mathcal{D}$ is dense in $\mathcal{H}$ and it follows that for $\phi, \, \psi \, \in \mathcal{D}$,

$$\begin{aligned} &[P_j, Q_l]^{\mathrm{w}}(\phi, \psi) = i\delta_{j,l}(\phi, \psi), \\ &[P_j, P_l]^{\mathrm{w}}(\phi, \psi) = [Q_j, Q_l]^{\mathrm{w}}(\phi, \psi) = 0. \end{aligned}$$

Let us define an abstract Dirac operator as follows. Let $(\mathcal{H}, \mathcal{D}, \mathbf{P}, \mathbf{Q})_3$ be the weak representation of the CCR with degree three. Let $\mathbf{A} = \{A_j\}_{j=1}^3$ and B be the bounded self-adjoint operators on a Hilbert space $\mathcal{K}$. Here $\mathbf{A} = \{A_j\}_{j=1}^3$ and B satisfy the canonical anti-commutation relations $\{A_j, A_l\} = 2\delta_{j,l}$, $\{A_j, B\} = 0$, $B^2 = I_{\mathcal{K}}$ where $I_{\mathcal{K}}$ is the identity operator on $\mathcal{K}$. The state Hilbert space space is defined by $\mathcal{H}_{\mathrm{Dirac}} = \mathcal{K} \otimes \mathcal{H}$. The free abstract Dirac operator is defiend by

$$H_0 = \sum_{j=1}^{3} A_j \otimes P_j + B \otimes M.$$

Here we assume the following condition.

> **(D.1)** P_j and P_l strongly commute for $1 \leq j \leq 3$, $1 \leq l \leq 3$. P_j, $1 \leq j \leq 3$, and M strongly commute.

Then it is seen that $H_0^2\Psi = \left(\mathbf{P}^2 + M^2\right)\Psi$ for $\Psi \in \mathcal{D}$. The abstract Dirac Operator with the Coulomb potential is defined by

$$H(\kappa) = H_0 + \kappa I_{\mathcal{K}} \otimes |\mathbf{Q}|^{-1},$$

where $\kappa \in \mathbf{R}$ is a parameter called the coupling constant. We assume that the following condition

> **(D.2)** It follows that $\mathcal{D} \subset \mathcal{D}(\mathbf{Q}|^{-1})$.

Then it follows from **(1)** in Theorem 1 that for $\psi \in \mathcal{D}$,

$$\|I_{\mathcal{K}} \otimes |\mathbf{Q}|^{-1}\psi\|^2 \leq 4\sum_{j=1}^{3} \|P_j\Psi\|^2 \leq 4\,\|H_0\Psi\|^2.$$

Hence by the Kato-Rellich theorem, we obtaine the following corollary.

Corollary 4 *Assume **(D.1)** and **(D.2)**. Then for* $|\kappa| < \frac{1}{2}$, $H(\kappa)$ *is essentially self-adjoint on* $\mathcal{D}$.

Acknowledgments

It is pleasure to thank assistant professor Akito Suzuki and associate professor Fumio Hiroshima for their advice and comments.

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